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CONTROLLABILITY OF NON-LINEAR SYSTEM*

Hector J. Sussman** and Velimir Jurdjevic***

1. Introduction

In this article we study the controllability of non-linear systems of the form

$$\frac{dx}{dt} = F(x, u). \qquad (*)$$

Our objective is to establish criteria in terms of $\underline{F}$ and its derivatives at a point $\underline{x}$ which will give qualitative information about the sets attainable from $\underline{x}$. The study is based primarily on the work of Chow [4] and Lobry [16], although it is similar in its approach to works by other authors in that it makes systematic use of differential geometry (for instance, see Hermann [8], [9], Haynes & Hermes [6], Brockett [2], etc.).

The state variable $\underline{x}$ is assumed to take values in an arbitrary real analytic manifold $\underline{M}$, rather than in $\tilde{R}^n$. We chose this generalization

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**Department of Mathematics, University of Chicago.

***Department of Mathematics, University of Toronto.

because it creates no essential new difficulties while, on the other hand, it allows for certain applications which are not commonly treated in control theory. For instance, when $\underline{M}$ is a Lie group, then the present results can be specialized to obtain more detailed controllability criteria. Control problems on Lie groups were first considered by R. W. Brockett in [2], and will be treated in a forthcoming paper by the authors.

Most of the recent studies on controllability of non-linear systems have essentially dealt with symmetric systems, i.e., systems of the form (*) with the property that $F(x,-u) = -F(x,u)$ (Hermann [9], Haynes and Hermes [6], Lobry [16]). As remarked by Lobry in [16], the consideration of symmetric systems often excludes interesting situations arising from mechanics. In these cases the system is of the form

$$\frac{dx}{dt} = A(x) + H(x)\cdot u.$$

A notable exception is the work by Lobry [17]. Lobry stated (and proved for the case of two vector fields in $\underset{\sim}{R}^3$) the result for non-symmetric systems that appears here as Theorem 3.1.

Our results apply to non-symmetric systems. We obtain some general information about the geometric structure of the attainable sets showing that they "practically" are submanifolds (see Theorems 4.4 and 4.5 for the precise statements). This information yields a complete answer to the problem of deciding when the sets attainable from a poi: $\underline{x}$ have a non-empty interior. The criteria obtained involve purely algebraic manipulations of $\underline{F}$ and its derivatives (of all orders) at the point x (see the Remark below).

In particular, our results contain those of Kučera [14]. In this connection we observe that our proofs are of interest even for the case treated by Kucera (see Sussman [21]).

We have omitted the consideration of non-autonomous systems; they can be treated analogously by the familiar procedure of reduction to an autonomous system (i. e., by considering the state variable to be defined in $\underline{M} \times \underset{\sim}{R}$).

The organization of the article is as follows: in section 2 we introduce notations and basic concepts; in addition, we quote some well-known basic results which will be used later. In section 3 we prove our main results in differential geometric terminology.

In section 4, we apply these results to control systems. We derive the algebraic criteria mentioned above (Corollaries 4. 6 and 4. 7) and we prove two "global results: we show that, for a large class of manifolds, accessibility (i. e. the property that, for any given $\underline{x}$, the set of points attainable from $\underline{x}$ has a nonempty interior) implies strong accessibility (i. e. that, for any given $\underline{x}$ and any given fixed positive $\underline{t}$, the set of points attainable from $\underline{x}$ at time $\underline{t}$ has a nonempty interior). We also show that, for a still larger class, including the Euclidean spaces, controllability implies strong accessibility.

Finally, section 5 contains examples. We show how our results can be used to derive the classical controllability criteria for the system

$$\frac{dx}{dt} = Ax + Bu .$$

We also derive the results of Kučera and indicate some generalizations.

Remark. An assumption that is made throughout the article is that F is an analytic function of x. This guarantees that all the information about the system is actually contained in F and its derivatives (of all orders) at a given point x. The analyticity assumption cannot be relaxed without destroying the theory (cf. Example 5. 3).

Another assumption that we make is that the trajectories of the system are everywhere defined. As opposed to the previous one, this assumption is not essential (except for the "global" Theorems 4. 9 and 4. 10). We use it, however, because it considerably simplifies all the proofs.

2. Preliminaries

We shall assume that the reader is familiar with the fundamental notions of differential geometry. All the definitions and basic concepts utilized in this paper can be found in standard books, (for instance, [1], [3], [7], [13] and [19]).

The following notations will be used throughout:

R--the set of real numbers.

R^n--n-dimensional Euclidean space.

M_x--the tangent space to the manifold M at the point x.

TM--the tangent bundle of the manifold M.

$V(M)$--the set of all analytic vector fields on the analytic manifold M.

We will regard $V(M)$ as a Lie algebra over the reals. For any X and Y in $V(M)$, we will denote the Lie product by

$[X, Y]$(i. e., $[X, Y] = XY - YX$). All the manifolds will be assumed to be paracompact. Recall that a submanifold of a paracompact manifold is paracompact. Also, a connected paracompact manifold is a countable union of compact sets. These facts imply (cf. Lobry [16], p 589):

LEMMA 2. 1. Let M be a (paracompact) manifold of dimension n. Let S be a k-dimensional connected submanifold of M. If $k < n$, then the set of points of S has an empty interior in M.

A subset D of $V(M)$ will be called *involutive* if, whenever X and Y belong to D, then $[X, Y]$ also belongs to D. A subalgebra of $V(M)$ is an involutive subspace. Let $D \subset V(M)$. An *integral manifold* of D is a connected submanifold S of M with the property that $S_x = \mathscr{L}(D(x))$ for every $x \in S$, where $D(x) = \{X(x) : X \in D\}$, and where $\mathscr{L}(D(x))$ is the subspace of M_x spanned by D(x). We state the following basic results about integral manifolds:

LEMMA 2. 2. Let D be an involutive subset of $V(M)$, and let $x \in M$. Then x is contained in a unique maximal integral manifold of D (here "maximal" means "maximal with respect to inclusion").

This result is classical if the dimension of $\mathscr{L}(D(x))$ is the same for each $x \in M$(Chevalley [3]). For a proof in the general case, see Lobry [16].

If $D \subset V(M)$, we denote the smallest subalgebra of $V(M)$ which contains D by $\mathscr{J}(D)$, and the maximal integral manifold of [illegible])) through x by $I(D, x)$. Recall that, if X is a vector field on M, then [illegible] an *integral curve* of X if α is a smooth mapping from a closed interval I, $I \subset \mathbb{R}$, into M such that

$\frac{d\alpha(t)}{dt} = X(\alpha(t))$ for all $t \in I$.

DEFINITION 2. 3. If D is a subset of $V(M)$, then an integral curve of D is a mapping α from a real interval $[t, t']$ into M such that there exist $t = t_0 < t_1 < \ldots < t_k = t$, and elements $X_1, \ldots, X_k$ of D with the property that the restriction of α to $[t_{i-1}, t_i]$ is an integral curve of X_i for each $i = 1, 2, \ldots, k$. We have the following elementary fact:

LEMMA 2. 4. Let $D \subset V(M)$. Let α: $[t_0, t_1] \to M$ be an integral curve of D, and let $a(t) = x$ for some $t \in [t_0, t_1]$. Then $\alpha(s) \in I(D, x)$ for all $s \in [t_0, t_1]$.

Proof. It is sufficient to consider the case when α is an integral curve of X, $X \in D$. For each maximal integral manifold S of $\mathscr{T}(D)$, let $J(S)$ be the set of all $s \in [t_0, t_1]$ such that $\alpha(s) \in S$. From the local existence and uniqueness of solutions of ordinary differential equations it follows that, if $s \in J(S)$, then there exists $r > 0$ such that $(s-r, s+r) \cap [t_0, t_1] \subset J(S)$. Thus, $J(S)$ is open relative to $[t_0, t_1]$. Since the maximal integral manifolds of $\mathscr{T}(D)$ are disjoint, we have that, for some maximal integral manifold S, $[t_0, t_1] \subset J(S)$. But $\alpha(t) \in I(D, x)$; therefore, our proof is complete.

Chow's theorem provides a partial converse to the above lemma. If $D \subset V(M)$, then D is symmetric if, whenever $X \in D$, $-X$ also belongs to D. We can now state Chow's theorem as follows:

LEMMA 2. 5. Let $D \subset V(M)$ be symmetric, and let $x \in M$. Then, for every $y \in I(D, x)$ there exists an integral curve α: $[0, T] \to M$ of D, with $T \geq 0$, such that $\alpha(0) = x$ and $\alpha(T) = y$.

In other words, every point of the maximal integral manifold of $\mathscr{T}(\underline{D})$ through $\underline{x}$ can be reached in positive time by following an integral curve of $\underline{D}$ having $\underline{x}$ as its initial point.

DEFINITION 2.6. Let $\underline{D} \subset \underline{V}(\underline{M})$, and let $\underline{x} \in \underline{M}$. If $T \geq 0$, then, for any $\underline{y} \in \underline{M}$, $\underline{y}$ is __D-reachable from x at time__ $\underline{T}$ if there exists an integral curve α of $\underline{D}$ defined on $[0, T]$ such that $\alpha(0) = x$ and $\alpha(T) = y$. The set of all $\underline{D}$-reachable points from $\underline{x}$ at time $\underline{T}$ is denoted by $L_x(D, T)$. The union of $L_x(D, t)$ for $0 \leq t < \infty$ (respectively for $0 \leq t \leq T$) is denoted by $\underset{\sim}{L}_x(D)$ (respectively $\underset{\sim}{L}_x(D, T)$).

3. Integrability of Families of Analytic Vector Fields

As an introduction to the general situation, we first considered the case when $\underline{D}$ is a symmetric subset of $\underline{V}(\underline{M})$. Chow's theorem can be utilized to obtain a necessary and sufficient condition for $\underset{\sim}{L}_x(D)$ to have a non-empty interior in $\underline{M}$. Let $\underline{n} = \dim \underline{M} = \dim \mathscr{T}(\underline{D})(\underline{x})$. Then $\underline{I}(\underline{D}, \underline{x})$ is an $\underline{n}$-dimensional submanifold of $\underline{M}$, and hence is open in $\underline{M}$. By Chow's theorem we have that $\underset{\sim}{L}_x(D) = I(D, x)$. We conclude that $\underset{\sim}{L}_x(D)$ is open in $\underline{M}$. Conversely (and without invoking the symmetry of $\underline{D}$) if $\dim \mathscr{T}(\underline{D})(\underline{x}) < \underline{n}$, then $\underline{I}(\underline{D}), \underline{x})$ is a connected submanifold of $\underline{M}$ of dimension less than $\underline{n}$; then from Lemma 2.1 it follows directly that $\underline{I}(\underline{D}, \underline{x})$ has an empty interior in $\underline{M}$. Since $\underset{\sim}{L}_x(D) \subset I(D, x)$, $\underset{\sim}{L}_x(D)$ also has an empty interior. Thus, if $\underline{D}$ is symmetric, a necessary and sufficient condition for $\underset{\sim}{L}_x(D)$ to have a non-empty interior in $\underline{M}$ is that $\dim \mathscr{T}(\underline{D})(\underline{x}) = \dim \underline{M}$. Moreover, this condition is necessary even in the non-symmetric case (Lobry [16]). We shall show that it is also sufficient. For this purpose we shall assume that the elements of $\underline{D}$ are complete--recall that a vector field $\underline{X}$ is __complete__ the integral curves of $\underline{X}$ are defined for all real $\underline{t}$ (cf. [13], p. 13).

THEOREM 3. 1. Let $\underline{M}$ be an $\underline{n}$-dimensional analytic manifold, and let $\underline{D} \subset \underline{V}(\underline{M})$ be a family of complete vector fields. A necessary and sufficient condition for $\underset{\sim}{L}_x(D)$ to have a non-empty interior in $\underline{M}$ is that $\dim \mathscr{I}(\underline{D})(\underline{x}) = \underline{n}$. Moreover, if this condition is satisfied, then for each $\underline{T} > 0$, the interior of $\underset{\sim}{L}_x(D, T)$ is dense in $\underset{\sim}{L}_x(D, T)$ (thus, in particular, $\underset{\sim}{L}_x(D, T)$ has a non-empty interior).

Proof. We already know that the condition of the theorem is necessary. So we assume that $\dim \mathscr{I}(\underline{D})(\underline{x}) = \underline{n}$, and we prove the second statement. Clearly, this will imply that $\underset{\sim}{L}_x(D)$ has a non-empty interior in $\underline{M}$. Without loss of generality we can assume that $\underline{D}$ is finite. Let $D = \{X_1, \ldots, X_k\}$. For each $i = 1, 2, \ldots, k$, let $\Phi_i(t, \cdot)$ be the one-parameter group of diffeomophisms induced by X_i(i. e., $t \to \Phi_i(t, y)$ is the integral curve of X_i which passes through $\underline{y}$ at $\underline{t} = 0$; the fact that it is defined for all real $\underline{t}$ follows from the completeness of X_i). If $\underline{m}$ is a natural number $\underset{\sim}{t} = (t_1, \ldots, t_m)$ is an element of $\underset{\sim}{R}^m$, and $\underset{\sim}{i} = (i_1, \ldots, i_m)$ is an $\underline{m}$-tuple of natural numbers between 1 and $\underline{k}$, then we denote the element $\Phi_{i_1}(t_1, \Phi_{i_2}(t_2, \ldots, \Phi_{i_m}(t_m, x) \ldots))$ by $\Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$. Let $\underline{\pm D}$ be the family of vector fields obtained from $\underline{D}$ by adjoining the vector fields $-X_1, \ldots, -X_k$ to $\underline{D}$. Then, $\underline{\pm D}$ is symmetric, and $\dim \mathscr{I}(\underline{\pm D})(x) = n$. From Chow's theorem we conclude that $\underset{\sim}{L}_x(\underline{\pm D})$ is open in $\underline{M}$. Clearly, the elements of $\underset{\sim}{L}_x(\underline{\pm D})$ are exactly those elements of $\underline{M}$ which are of the form $\Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$ for some $\underline{m}$, some $\underline{m}$-tuple $\underset{\sim}{i}$, and some $\underset{\sim}{t} \in \underset{\sim}{R}^m$. For each $\underset{\sim}{i}$, and for each natural number $\underline{N} > 0$, let $\underline{A}(\underset{\sim}{i}, \underline{N})$ be the set of all points of $\underline{M}$ of the form $\Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$, where $\|\underset{\sim}{t}\| \leq \underline{N}$ (here, $\|\underset{\sim}{t}\| = |t_1| + \ldots + |t_m|$). Since $\underline{A}(\underset{\sim}{i}, \underline{N})$ is the image of the compact set $\{\underset{\sim}{t} : \|\underset{\sim}{t}\| \leq N\}$ under the continuous mapping $\underset{\sim}{t} \to \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$, we have that $\underline{A}(\underset{\sim}{i}, \underline{N})$ is compact. Also, since $\underset{\sim}{L}_x(\pm \underline{D})$ is the union of the sets $\underline{A}(\underset{\sim}{i}, \underline{N})$ (taken over $\underline{m}$, $\underset{\sim}{i}$ and $\underline{N}$), it follows from the category theorem that, for some $\underset{\sim}{i}$ and $\underline{N}$, the set $\underline{A}(\underset{\sim}{i}, \underline{N})$ has a non-empty interior in $\underline{M}$. For such an $\underset{\sim}{i}$, let $F: \underset{\sim}{R}^m \to M$ be defined by $F(\underset{\sim}{t}) = \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$. Then $\underline{F}$ is an analytic mapping whose image has a non-empty interior in $\underline{M}$. By Sard's theorem (Sternberg [19]), the differential $dF_{\underset{\sim}{t}}$ of $\underline{F}$ at $\underset{\sim}{t}$ must have rank $\underline{n}$ for some $\underset{\sim}{t} \in \underset{\sim}{R}^m$. Since $dF_{\underset{\sim}{t}}$ depends analytically on $\underset{\sim}{t}$, it follows that the set $\Omega^{\#} = \{\underset{\sim}{t} : \underset{\sim}{t} \in \underset{\sim}{R}^m, \text{ rank } \underline{dF}_{\underset{\sim}{t}} < \underline{n}\}$ has an empty interior. Let $\Omega = \underset{\sim}{R}^m - \Omega^{\#}$. Then Ω is open and dense in $\underset{\sim}{R}^m$.

Let $\underline{T} > 0$, and let $y \in \underset{\sim}{L}_x(D, T)$. We now show that $\underline{y}$ is in the closure of the interior of $\underset{\sim}{L}_x(D, T)$. It is clearly sufficient to assume that $\underline{y} \in L_x(D, t)$, where $0 \leq t < T$ (for each point of $\underset{\sim}{L}_x(D, T)$ is in the closure of $\bigcup \{L_x(D, t) : 0 \leq t < T\}$). Let $y = \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, x)$ where $\underset{\sim}{j} = (j_1, \ldots, j_p)$, $\underset{\sim}{s} = (s_1, \ldots, s_p)$, $s_1 > 0, \ldots, s_p > 0$, and $s_1 + \ldots + s_p = t$. Let $\underline{U} = \Omega \cap \{\underset{\sim}{t} : \|\underset{\sim}{t}\| < T - t\} \cap \{\underset{\sim}{t} : t_1 > 0, \ldots, t_m > 0\}$. $\underline{U}$ is open, and its closure contains the original $\underset{\sim}{0}$ of $\underset{\sim}{R}^m$. Since $dF_{\underset{\sim}{t}}$ has rank $\underline{n}$ at each point $\underset{\sim}{t} \in \underline{U}$, it follows that $\underline{F}(\underline{U})$ is open. Let $V = \{\Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, F(\underset{\sim}{t})) : \underset{\sim}{t} \in U\}$. $\underline{V}$ is the image of $\underline{F}(\underline{U})$ under the diffeomorphism $\underline{z} \to \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, \underline{z})$; therefore, $\underline{V}$ is open in $\underline{M}$ and, moreover, every element of $\underline{V}$ is $\underline{D}$-reachable from $\underline{x}$ at time $\|\underset{\sim}{s}\| + \|\underset{\sim}{t}\| = t + \|\underset{\sim}{t}\| < T$ (here we use essentially the fact that $t_1, \ldots, t_m$ are non-negative). It remains to be shown that $\underline{y}$ belongs to the closure of $\underline{V}$. Let $\{\underset{\sim}{t}_q\}$ be a sequence of elements of $\underline{U}$ which converges to $\underset{\sim}{0}$. Then

$$\lim \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, F(\underset{\sim}{t}_{\underline{q}})) = \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, F(\underset{\sim}{0})) = \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, x) = y.$$

This completes the proof of the theorem.

We now want to state an analogous theorem for the sets $L_x(D, T)$. For this purpose, we shall introduce a Lie subalgebra $\mathscr{T}_0(D)$ of $\mathscr{T}(D)$ which will be related to these sets in the same way as $\mathscr{T}(D)$ is related to the sets $\underset{\sim}{L}_x(D, T)$. The aim of the following informal remarks is to motivate our definition of $\mathscr{T}_0(D)$. We shall ignore the fact that time has to be positive. Moreover, we shall assume, for simplicity, that $\underline{D}$ consists of three vector fields X_1, X_2 and X_3. Let Φ_1, Φ_2 and Φ_3 be the corresponding one-parameter groups. It is clear that $\mathscr{T}(D)$ has the following "geometric interpretation": $\mathscr{T}(D)(x)$ is, for each $x \in M$, the set of all limiting directions of curves through $\underline{x}$ that are entirely contained in $\underset{\sim}{L}_x(D)$. Thus, for instance, if $i=1, 2, 3$, then all the points in the curve $t \to \Phi_i(t, x)$ are attainable from $\underline{x}$ (recall that we are forgetting about positivity), and this is reflected in the fact that $X_i(x)$ belongs to $\mathscr{T}(D)(x)$. Similarly, the curves $\alpha_{ij}(t) = \Phi_i(-t, \Phi_j(-t, \Phi_i(t, \Phi_j(t, x))))$ are also contained in $\underset{\sim}{L}_x(D)$. By the well known geometric interpretation of the Lie bracket (cf. Helgason [7], p. 97), the limiting direction of α_{ij} is $[X_i, X_j](x)$ (after a reparametrization). Thus, it is clear why $[X_i, X_j]$ belongs to $\mathscr{T}(D)$. Obviously, a similar argument works for the brackets of higher order. The geometrical meaning of $\mathscr{T}(D)$ is now obvious.

If $\mathscr{T}_0(D)$ is going to play the desired role it is clear that $\mathscr{T}_0(D)(x)$ will have to be the set of all limiting directions of curves γ through $\underline{x}$ such that $\gamma(t)$ is "attainable from $\underline{x}$ in zero units of time" for all $\underline{t}$. Notice that the curves $\alpha_{ij}(t)$ of the preceding paragraph have this property. Indeed, $\alpha(t)$ can be reached from $\underline{x}$ by "moving forward" in time 2t units, and then "backward" another 2t units. This shows that the vector fields $[X_i, X_j]$ are reasonable candidates for membership in $\mathscr{T}_0(D)$. A similar argument applies to higher order brackets, such as $[X_i, [X_j, X_k]]$, etc. On the other hand, a vector field such as X_i should not be included in $\mathscr{T}_0(D)$ by definition, because we do not know whether the points $\phi_i(t, x)$, $t \neq 0$, can be reached from $\underline{x}$ in 0 units of time (but, of course, it may happen that some X_i will belong to $\mathscr{T}_0(D)$ anyhow; for instance, we could have $X_1 = [X_2, X_3]$). However, the vector fields $X_i - X_j$ will have to be included, because $(X_i - X_j)(x)$ is the limiting direction of the curve $t \to \phi_j(-t, \phi_i(t, x))$. In other words, the subspace generated by the differences $X_i - X_j$ will have to be included in $\mathscr{T}_0(D)$. This subspace can also be defined as the set of all linear combinations $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (that all the differences $X_i - X_j$ are linear combinations of this type is trivial; conversely, if $Y = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then $Y = \lambda_1 X_1 + \lambda_2 X_2 + (-\lambda_1 - \lambda_2) X_3$, i.e., $Y = \lambda_1(X_1 - X_3) + \lambda_2(X_2 - X_3)$).

We conclude that the reasonable candidates for membership in $\mathscr{T}_0(D)$ are: (i) all the brackets $[X_i, X_j]$, $[X_i, [X_j, X_k]]$, etc., and (ii) all the sums $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, where $\Sigma \lambda_i = 0$. Notice that the subset generated by (i) is clearly the derived algebra of $\mathscr{T}(D)$ (by definition, the derived algebra of a Lie algebra $\underline{L}$ is the subalgebra $\underline{L}'$ of $\underline{L}$ generated by all the brackets $[X, Y]$, $X \in L$, $Y \in L$; it is easy to check that L' is in fact an ideal of $\underline{L}$; cf. Helgason [7], p. 133.

We now return to our formal development. Let $\mathscr{T}'(D)$ denote the derived algebra of $\mathscr{T}(D)$. Motivated by the previous remarks, we define $\mathscr{T}_0(D)$ to be the set of all sums $X + Y$, where X is a linear combination

$$\sum_{i=1}^{p} \lambda_i X_i \text{ with } X_1, \ldots, X_p \in D \text{ and } \Sigma \lambda_i = 0,$$

and where $Y \in \mathscr{T}'(D)$. It is obvious that $\mathscr{T}_0(D)$ is an ideal of $\mathscr{T}(D)$. One shows easily that $\mathscr{T}(D)$ is the set of all vector fields of the form

$$\sum_{i=1}^{p} \lambda_i X_i + Y$$

where $X_1, \ldots, X_p$ belong to D, Y belongs to $\mathscr{T}'(D)$, and $\lambda_1, \ldots, \lambda_p$ are reals (but $\lambda_1 + \ldots + \lambda_p$ need not be zero). From this it follows immediately that $\mathscr{T}_0(D)$ is a subspace of $\mathscr{T}(D)$ of codimension zero or one. The codimension will be zero if and only if some $X \in D$ belongs to $\mathscr{T}_0(D)$ (in which case every $X \in D$ will belong to $\mathscr{T}_0(D)$). Similarly for each $x \in M$, if $k = \dim \mathscr{T}(D)(x)$, then the dimension of $\mathscr{T}_0(D)(x)$ will either be k or k-1.

We shall also be interested in associating to each $D \subset V(M)$, a set D^* of vector fields in the manifold $M \times \underset{\sim}{R}$. Recall that the tangent space to $M \times R$ at a point (x, r) ($x \in M$, $r \in \underset{\sim}{R}$) is identified, in a natural way, to the direct sum $M_x \oplus \underset{\sim}{R}_r$. If $x \in V(M)$, $Y \in V(\underset{\sim}{R})$, we define the vector field $X \oplus Y \in V(M \times \underset{\sim}{R})$ by

$$(X \oplus Y)(x, r) = (X(x), Y(r)).$$

The set D^* is defined to be the set of all vector fields $X \oplus \frac{\partial}{\partial t}$, where $X \in D$, and where $\frac{\partial}{\partial t}$ is the "canonical" vector field on $\underset{\sim}{R}$ ($(\frac{\partial}{\partial t} f)(r) = \frac{df}{dt}(r)$). Using the identity $[X \oplus X', Y \oplus Y'] = [X, Y] \oplus [X', Y']$, one shows easily that $\mathscr{T}'(D^*)$ is the set of all vector fields of the form $X \oplus \underset{\sim}{0}$, where $X \in \mathscr{T}'(D)$ and $\underset{\sim}{0}$ is the zero vector field. Therefore, $\mathscr{T}(D^*)$ is the set of vector fields of the form

$$\sum_{i=1}^{p} \lambda_i (X_i \oplus \frac{\partial}{\partial t}) + Y \oplus \underset{\sim}{0} \qquad (\#)$$

where $X_1, \ldots, X_p$ belong to D, $Y \in \mathscr{T}'(D)$, and $\lambda_1, \ldots, \lambda_p$ are scalars.

THEOREM 3.2. Let $\underline{M}$ be an analytic $\underline{n}$-dimensional manifold, and let $\underline{D}$ be a family of complete analytic vector fields on M. Let $\underline{x} \in \underline{M}$, and let $\underline{T} > 0$. Then $L_x(D,T)$ has a non-empty interior in $\underline{M}$ if and only if $\dim \mathscr{T}_0(D)(x) = n$. Moreover, in this case, the interior of $L_x(D,T)$ is dense in $L_x(D,T)$.

Proof. The main idea in this proof is to modify our problem so that we can "keep track" of the time elapsed while we move along an integral curve of $\underline{D}$. We shall then apply Theorem 3.1 to the modified system. We shall work in the manifold $\underline{M} \times \underset{\sim}{R}$. As in the preceding paragraphs, we let the family $\underline{D}^*$ of vector fields on $\underline{M} \times \underset{\sim}{R}$ be defined by $D^* = \{X \oplus \frac{\partial}{\partial t} : X \in L\}$. It is clear that there is a one-to-one correspondence between integral curves α of $\underline{D}$ such that $\alpha(0) = x$, and integral curves $\beta(\underline{D}^*)$ such that $\beta(0) = (x, 0)$. This correspondence is given by assigning to each curve α the curve $t \to (\alpha(t), t)$. It follows that $y \in L_x(D,T)$ if and only if $(y, T) \in \underset{\sim}{L}_{(x,0)}(D^*, T)$. We show that $L_x(D,T)$ has a non-empty interior in $\underline{M}$ if and only if $\underset{\sim}{L}_{(x,0)}(D^*)$ has a non-empty interior in $\underline{M} \times \underset{\sim}{R}$. Assume that $L_x(D,T)$ has a non-empty interior in $\underline{M}$, and let $\underline{V}$ be a non-empty open set such that $V \subset L_x(D,T)$. Let $\underline{X} \in \underline{D}$, and let Φ be the one-parameter group of diffeomorphisms of $\underline{M}$ generated by $\underline{X}$. Consider the mapping $F : V \times \underset{\sim}{R} \to M \times \underset{\sim}{R}$ defined by $F(v,t) = (\Phi(t,v), T + t)$. It is immediate that the differential of $\underline{F}$ has rank $\underline{r} + 1$ everywhere. Therefore $\underline{F}$ maps open sets onto open sets.

Since $F(VX(0,\infty)) \subset \underset{\sim}{L}_{(x,0)}(D^*)$, we conclude that $\underset{\sim}{L}_{(x,0)}(D^*)$ has a non-empty interior in $M X \underset{\sim}{R}$.

To prove the converse, assume that $\underset{\sim}{L}_{(x,0)}(D^*)$ has a non-empty interior in $\underline{M} X \underset{\sim}{R}$. By Theorem 3.1, for each $\underline{t}$ with $0 < \underline{t} < \underline{T}$, $\underset{\sim}{L}_{(x,0)}(D^*, t)$ has a non-empty interior in $\underline{M}X\underset{\sim}{R}$. Let $\underline{V}$ be a non-empty open subset of $\underline{M}$, and let $\underline{W}$ be a non-empty open subset of $\underset{\sim}{R}$ such that $\underline{V}X\underline{W} \subset \underset{\sim}{L}_{(x,0)}(D^*, t)$. Let $\underline{s} \in \underline{W}$. Since $\underline{V} X \{\underline{s}\} \subset \underset{\sim}{L}_{(x,0)}(D^*, t)$, we conclude that $V \subset L_x(D, s)$. Let $\underline{X} \in \underline{D}$, and let Φ be the corresponding one-parameter group on $\underline{M}$. Denote the mapping $y \to \Phi(T-s, y)$ by $\underline{G}$. Then $\underline{G}(\underline{V})$ is open. Since $\underline{G}(\underline{V})$ is contained in $L_x(D, T)$, it follows that $L_x(D, T)$ has a non-empty interior.

We conclude from Theorem 3.1 that $L_x(D, T)$ has a nonempty interior if and only if $\dim \mathscr{T}(D^*)(x, 0) = n + 1$. To complete the proof of the first part of our statement, we must show that this last condition holds if and only if $\dim \mathscr{T}_0(D)(x) = \underset{\cdot}{n}$. We recall, from the remarks preceding this proof, the fact that every $X^* \in \mathscr{T}(D^*)$ can be expressed as

$$(\#) \quad X^* = \sum_{i=1}^{p} \lambda_i \left(X_i \oplus \frac{\partial}{\partial t}\right) + Y \oplus \underset{\sim}{0} \text{ where } X_1, \ldots, X_p \text{ belong to } \underline{D}$$

and $Y \in \mathscr{T}'(D)$. Now assume that $\dim \mathscr{T}(D^*)(x, 0) = n + 1$. Let $v \in M_x$. Then $(v, 0)$ must belong to $\mathscr{T}(D^*)(x, 0)$, so that $(v, 0) = X^*(x, 0)$, where $X^* \in \mathscr{T}(D^*)$. Then formula (#) holds for suitable λ_i, X_i, Y. Therefore

$$v = (\Sigma \lambda_i X_i + Y)(x),$$

and

$$\underset{\sim}{0} = \Sigma \lambda_i \frac{\partial}{\partial t}(0).$$

The last equality implies that $\Sigma\lambda_i = 0$, so that the vector field $\Sigma\lambda_i X_i + Y$ belongs to $\mathscr{T}_0(D)$. Thus $v \in \mathscr{T}_0(D)(x)$. We have shown that $M_x \subseteq \mathscr{T}_0(D)(x)$. Therefore the dimension of $\mathscr{T}_0(D)(x)$ is $\underline{n}$. Conversely, let $\dim \mathscr{T}_0(D)(x) = n$. Let $v \in M_x$. Then $v \in \mathscr{T}_0(D)(x)$, so that

$$v = (\Sigma\lambda_i X_i + Y)(x),$$

where the X_i belong to D, $Y \in \mathscr{T}'(D)$ and $\Sigma\lambda_i = 0$. Therefore,

$$\begin{aligned}(v, 0) &= ((\Sigma\lambda_i X_i + Y) \oplus (\Sigma\lambda_i)\frac{\partial}{\partial t})(x, 0)\\ &= (\Sigma\lambda_i(X_i \oplus \frac{\partial}{\partial t}) + Y \oplus \underset{\sim}{0})(x, 0).\end{aligned}$$

This shows that $(v, 0)$ belongs to $\mathscr{T}(D^*)(x, 0)$. Pick an $X \in D$. Then $X \ominus \frac{\partial}{\partial t}(x, 0)$ belongs to $D^*(x, 0)$ by definition, and $X \oplus \underset{\sim}{0}$ $(x, 0)$ belongs to $\mathscr{T}(D^*)(x, 0)$ by the previous remarks. Therefore $(\underset{\sim}{0}, \frac{\partial}{\partial t}(0))$ belongs to $\mathscr{T}(D^*)(x, 0)$. We have thus shown that $\mathscr{T}(D^*)(x, 0)$ contains all the vectors $(v, 0)$, $v \in M_x$, and also the vector $(\underset{\sim}{0}, \frac{\partial}{\partial t}(0))$. Therefore $\mathscr{T}(D^*)(x, 0) = (M \times \underset{\sim}{R})_{(x, 0)}$, so that $\dim \mathscr{T}(D^*)(x, 0) = n + 1$ as stated.

We now prove the second part of the theorem. As we remarked earlier, there is no loss of generality in assuming that $\underline{D}$ is finite. Let $y \in L_x(D, T)$. Using the notations of the proof of Theorem 3.1, let $y = \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$, where $\underset{\sim}{i} = (i_1, \ldots, i_m)$, and where $\underset{\sim}{t} \in \underset{\sim}{R}^m$ is such that $t_i > 0$ for $i = 1, \ldots, m$ and $\|\underset{\sim}{t}\| = T$. Let $\{s_k\} \subset (0, t_m)$ be such that $\lim_{k \to \infty} s_k = 0$. Since our condition for $L_x(D, T)$ to have a non-empty interior is independent of $\underline{T}$, we conclude that $L_x(D, t)$ has a non-empty interior for all $\underline{t} > 0$. In particular, for each $\underline{k} > 0$, there exists x_k which belongs to the interior of $L_x(D, s_k)$. Let $\underset{\sim}{t}_k = (t_1, \ldots, t_{m-1},$

$t_m - s_k$), and let $y_k = \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}_k, x_k)$. For each $\underline{k} > 0$, y_k belongs to $L_x(D,T)$; since $\Phi_{\underset{\sim}{i}}$ is a diffeomorphism, y_k is the interior of $L_x(D,T)$. Also, $x_k \to x$ as $k \to \infty$ because $\underline{D}$ is finite and $s_k \to 0$. Since $\Phi_{\underset{\sim}{i}}$ is continuous in both variables, and since $\underset{\sim}{t}_k \to \underset{\sim}{t}$, we have that $y_k \to y$, and our theorem is proved.

The results of the previous theorems can be utilized to obtain information about the sets $\underset{\sim}{L}_x(D,T)$ and $L_x(D,T)$, even when $\dim \mathscr{T}(\underline{D})(\underline{x}) < \underline{n}$.

THEOREM 3.3. Let $D \subset V(M)$ be a family of complete vector fields. Then, for each $T > 0$, the set $\underset{\sim}{L}_x(D,T)$ is contained in $I(D,x)$. Moreover, in the topology of $I(D,x)$, the interior of $\underset{\sim}{L}_x(D,T)$ is dense in $\underset{\sim}{L}(D,T)$. $L_x(D,T)$ has a non-empty interior in $I(D,x)$ if and only if $\dim \mathscr{T}_0(D(x)) = \dim \mathscr{T}(D)(x)$ and, in this case, the interior of $L_x(D,T)$ is dense in $L_x(D,T)$.

Proof. If $\underline{X} \in \mathscr{T}(D)$, then $\underline{X}$ is tangent to $I(D,x)$. Thus, there is a well-defined restriction $\underline{X}\#$ of $\underline{X}$ to $\underline{I}(\underline{D},\underline{x})$. We denote the set of all such restrictions of elements of D by $D\#$. Since $[\underline{X},\underline{Y}]\# = [\underline{X}\#,\underline{Y}\#]$, it follows that $\mathscr{T}(\underline{D})\# = \mathscr{T}(\underline{D}\#)$. Analogously, we have that $\mathscr{T}_0(D)\# = \mathscr{T}_0(D\#)$. If we now apply the previous theorems to the family $D\#$ of vector fields in $I(D,x)$, we get all the conclusions of the theorem.

COROLLARY 3.4. Let $\underline{S}$ be a maximal integral manifold of $\mathscr{T}(\underline{D})$. Then the dimension of $\mathscr{T}_0(\underline{D})(\underline{x})$ is the same for all $\underline{x} \in \underline{S}$.

Proof. If $\dim \mathscr{T}(D)(x) = k$ then, for each $\underline{x} \in \underline{S}$, the dimension of $\mathscr{T}_0(D)(x)$ is either $\underline{k}$ or $\underline{k}-1$. We show that, if $\dim \mathscr{T}_0(D)(y) = k-1$ for some $\underline{x} \in \underline{S}$, then $\dim \mathscr{T}_0(D)(y) = k-1$ for all $\underline{y} \in \underline{S}$. Let Ω be a non-empty, open (relative to $\underline{S}$) subset of $\underset{\sim}{L}_x(D)$ (this is possible by Theorem 3.3). We first show that, if $\underline{y} \in \Omega$, then $\dim \mathscr{T}_0(D)(y) = k-1$. If this were not the case, then necessarily $\dim \mathscr{T}_0(D)(y) = k$. Then $L_y(D,t)$ would have a non-empty interior in $\underline{S}$ for all $\underline{t} > 0$. This would imply that $L_x(D, t)$ has a non-empty interior in $\underline{S}$. But by our assumption this is impossible. Thus, $\dim \mathscr{T}_0(D)(y) = \underline{k} - 1$ for all $\underline{y} \in \Omega$. Since $\underline{S}$ is connected, and Ω is open in $\underline{S}$, we have that $\dim \mathscr{T}_0(D)(y) = k-1$ for all $\underline{y} \in \underline{S}$; therefore, our statement is proved.

We now proceed to study the case when $\dim \mathscr{T}_0(D)(x) = \dim \mathscr{T}(D)(x) - 1$. We begin by proving some preliminary lemmas.

LEMMA 3.5. Let $\underline{D} \subset \underline{V}(\underline{M})$ be a family of complete vector fields. If $\underline{X} \in \underline{D}$, let $\{\Phi_t\}$ be the one-parameter group generated by $\underline{X}$. Then, for every $\underline{x} \in \underline{M}$, and every $\underline{t} \in \underset{\sim}{R}$ the differential $d\Phi_t$ maps $\mathscr{T}_0(D)(x)$ onto $\mathscr{T}_0(D)(\Phi_t(x))$.

Proof. We first show that for every $\underline{y} \in \underline{M}$ there is an $\underline{r} > 0$ such that, if $v \in \mathscr{T}_0(D)(y)$, then $d\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(y))$ for all $\underline{t}$ with $|t| < r$. It is sufficient to show that for every $\underline{y} \in \underline{M}$ and every $v \in \mathscr{T}_0(D)(y)$ there exists an $\underline{r} > 0$ such that $d\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(y))$ for all t with $|t| < r$. Let $\underline{y} \in \underline{M}$, and let $v \in \mathscr{T}_0(D)(y)$. If $v = Y(y)$ for some $Y \in \mathscr{T}_0(D)$, then an easy computation shows that there exists a neighborhood of $\underline{t} = 0$ such that $d\Phi_t(v) = \Sigma_{i=1}^{\infty} \frac{(-1)^i}{i!} [X^{(i)}, Y](\Phi_t(y))t^i$ for all $\underline{t}$ in this neighborhood, where $[X^{(0)}, Y] = Y$, and $[X^{(n)}, Y] = [X, [X^{(n-1)}, Y]]$ for $n = 1, 2, \ldots$ Since each term of the above series belongs to $\mathscr{T}_0(D)(\Phi_t(y))$, we have that $d\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(y))$ for $\underline{t}$ sufficiently small. Also, for such $\underline{t}$ we have that $d\Phi_t(\mathscr{T}_0(D)(y)) = \mathscr{T}_0(D)(\Phi_t(y))$; this is so because $d\Phi_t$ is one-to-one, and $\dim \mathscr{T}_0(D)(y) = \dim \mathscr{T}_0(D)(\Phi_t(y))$ (Corollary 3.4). It follows easily that the set of all $\underline{t}$ such that $d\Phi_t(\mathscr{T}_0(D)(x)) = \mathscr{T}_0(D)(\Phi_t(x))$ is both open and closed. If $v \in \mathscr{T}_0(D)(x)$, we can conclude that $\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(v))$ for all t. This completes our proof.

As we remarked earlier, $\mathscr{T}_0(\underline{D})$ is a subalgebra of $\mathscr{T}(D)$. We will denote the maximal integral manifold of $\mathscr{T}_0(\underline{D})$ through $\underline{x}$ by $I_0(D, x)$. If $\underline{X} \in \underline{D}$ then, by the previous lemma, $\Phi_t(I_0(D, x))$ is a maximal integral manifold of $\mathscr{T}_0(D)$.

LEMMA 3.6. Let $\underline{D} \subset \underline{V}(\underline{M})$ be a family of complete vector fields. Let $\underline{X}$ and $\underline{Y}$ be elements of $\underline{D}$, and let $\{\Phi_t\}$ and $\{\Psi_t\}$ be their corresponding one-parameter groups. If $\underline{S}$ is a maximal integral manifold of $\mathscr{T}_0(D)$ then, for any $\underline{t} \in \underset{\sim}{R}$, $\Phi_t(S) = \Psi_t(S)$.

Proof. Let $\underline{X}, \underline{Y}$, Φ_t, Ψ_t and $\underline{S}$ satisfy the conditions of the lemma. Let P be the maximal integral manifold of $\mathscr{T}(D)$ which contains $\underline{S}$. If $\dim \underline{P} = \dim \underline{S}$, then $\underline{S} = \underline{P}$, and $\Phi_t(S) = S = \Psi_t(S)$. Assume that $\dim S = k = \dim(P) - 1$. We first show that there is an $\underline{r} > 0$ such that $\Phi_t(S) = \Psi_t(S)$ whenever $|t| < r$. Let $\underline{x} \in \underline{S}$. The mapping $(s, t) \to \Phi_t(s)$ has rank $\underline{k} + 1$ at $(\underline{x}, 0)$. Let Ω be a neighborhood of $\underline{x}$ in $\underline{S}$, and let $\delta > 0$ be such that this mapping, restricted to $\Omega \times (-\delta, \delta)$ is a diffeomorphism onto an open subset $\Omega\#$ of $\underline{P}$. If $\underline{y} \in \Omega\#$, let $\underline{s}(\underline{y})$ and $\underline{f}(\underline{y})$ be such that $\Phi_{f(y)}(s(y)) = y$. Clearly, $\underline{f}$ is analytic in $\Omega\#$, and $f(y) = 0$ if and only if $\underline{y} \in \Omega$. Moreover, $Xf \equiv 1$ in $\Omega\#$. For every $\underline{t}$ such that $|t| < \delta$, the set $\Phi_t(\Omega)$ is an integral manifold of $\mathscr{T}_0(\underline{D})$. The vector field $\underline{Y} - \underline{X}$ is tangent to $\Phi_t(\Omega)$ and, since $\underline{f}$ is constant on $\Phi_t(\Omega)$, it follows that $Yf \equiv Xf$ on $\Phi_t(\Omega)$. Since $\Omega\#$ is the union of the sets $\Phi_t(\Omega)$ over $-\delta < t < \delta$, we conclude that $Yf \equiv Xf \equiv 1$ on $\Omega\#$. Let $r > 0$ be such that the curve $t \to \Phi_{-t}(\Psi_t(x))$, defined on $-r < t < r$, is contained in $\Omega\#$. In addition, let $r < \delta$. Let $g(t) = f(\Phi_{-t}(\Psi_t(x)))$. Then $\underline{g}$ is analytic in $(-r, r)$, and moreover $\underline{g}(t) = f(\Psi_t(x)) - t$. We have that $q' = (Yf)(\Psi_t(x)) - 1 \equiv 0$ and, since $g(0) = 0$ it follows that $g \equiv 0$ on $(-r, r)$. But this means that $\Phi_{-t}(\Psi_t(x)) \in \Omega$ for all $t \in (-r, r)$. Hence, if $|t| < r$, the manifold

$\Phi_{-t}(\Psi_t(S))$ intersects $\underline{S}$. Since $\Phi_{-t}(\Psi_t(S))$ is a maximal integral manifold of $\mathscr{T}_0(D)$, it follows that $\Phi_{-t}(\Psi_t(S)) = S$, and that $\Psi_t(S) = \Phi_t(S)$. Let $\underline{A}$ be the set of all $\underline{t}$ such that $\Phi_\tau(S) = \Psi_\tau(S)$ for all τ in a neighborhood of $\underline{t}$. Then $\underline{A}$ is obviously open, and we have shown that $0 \in \underline{A}$. It follows easily from the preceding argument that $\underline{A}$ is closed. Therefore, $\Phi_t(S) = \Psi_t(S)$ for all real $\underline{t}$, and our proof is complete.

According to the above lemma, if $\underline{D} \subset \underline{V}(\underline{M})$ and if $\underline{x} \in \underline{M}$, then the manifold $\Phi_t(I_0(D,x))$ depends only on $\underline{t}$, and not on the particular choice of $\underline{X}$. We shall denote this manifold by $I_0^t(D,x)$. It is clear that $I_0^t(D,x)$ could be defined as the maximal integral manifold of $\mathscr{T}_0(D)$ passing through y, where y is an arbitrary point of $L_x(D,t)$.

Finally, we prove a factorization property of maps that will be utilized several times.

LEMMA 3.7. Let $\underline{E}$ be a locally convex vector space, let $\underline{K} \subset \underline{E}$, and let $\underline{C}$ be a convex dense subset of $\underline{K}$. Let $F : K \to I(D,x)$ be a continuous mapping such that $\underline{F}(\underline{C})$ is contained in a maximal integral manifold $\underline{S}$ of $\mathscr{T}_0(\underline{D})$. Then $\underline{F}(\underline{K})$ is contained in $\underline{S}$, and $\underline{F}$, as a mapping from $\underline{K}$ into $\underline{S}$, is continuous.

Proof. If $\dim S = \dim I(D,x)$, then $S = I(D,x)$, and the conclusion follows trivially. Therefore, we shall assume that $\dim S = \dim I(D,x) - 1$.

Let $\underline{k} \in \underline{K}$, let $\underline{X} \in \underline{D}$, and let $\{\Phi_t\}$ be the one-parameter group induced by $\underline{X}$. Then, as in the proof of Lemma 3.6, we can find a neighborhood Ω of $\underline{F}(\underline{k})$ in $I_0(D, \underline{F}(\underline{k}))$, and a positive number δ, such that the mapping $(s,t) \to \Phi_t(s)$ is a diffeomorphism of $\Omega \times (-\delta, \delta)$ onto an open subset $\Omega^{\#}$ of $I(D,x)$. Let $\underline{U}$ be an open convex neighborhood of $\underline{k}$ such that $\underline{F}(\underline{U} \cap \underline{K}) \subset \Omega^{\#}$. For each $\underline{t} \in (-\delta, \delta)$, the set $\Phi_t(\Omega)$ is an integral manifold of $\mathcal{T}_0(D)$; therefore, if $\Phi_t(\Omega)$ intersects $\underline{S}$, then $\Phi_t(\Omega)$ is contained and open in $\underline{S}$. Let $A = \{t : |t| < \delta,\ \Phi_t(\Omega) \subset S\}$. It follows that $\underline{S} \cap \Omega^\#$ is the union of the sets $\Phi_t(\Omega)$, $\underline{t} \in \underline{A}$. These sets are mutually disjoint and, since $\underline{S}$ is separable, it follows that A is at most countable. Let $y \to (s(y), f(y))$ be the inverse in $\Omega^\#$ of the map $(s, t) \to \Phi_t(s)$. Then the function $\underline{g}$ defined in $\underline{U} \cap \underline{K}$ by $g(m) = f(F(m))$ is continuous. Since $F(U \cap C) \subset S \cap \Omega^\#$, we conclude that $\underline{g}(\underline{m}) \in \underline{A}$ for all $\underline{m} \in \underline{U} \cap \underline{C}$. But $\underline{A}$ is at most countable, and $\underline{U} \cap \underline{C}$ is convex; therefore $\underline{g}$ is constant on $\underline{U} \cap \underline{C}$. Since $\underline{U} \cap \underline{C}$ is dense in $\underline{U} \cap \underline{K}$, we have that $\underline{g}$ is constant on $\underline{U} \cap \underline{K}$. Obviously $g(k) = 0$, and therefore $g(m) = 0$ for all $\underline{m} \in \underline{U} \cap \underline{K}$; thus $\underline{F}(\underline{m}) \in \Omega$. This shows that Ω contains a point of $\underline{S}$; hence $\Omega \subset \underline{S}$, and $\underline{F}(\underline{k}) \in \underline{S}$. This proves the first part of the lemma.

To prove the second part, let $\{k_n\} \subset K$ converge to $\underline{k}$. Since $\underline{F}$ is continuous, $F(k_n) \to F(k)$. For large $\underline{n}$, $s(F(k_n))$ is defined. Since $\underline{s}$ is continuous, $s(F(k_n))$ converges to $s(F(k))$ in $\underline{S}$. But $g(k_n) = 0$, and therefore $s(F(k_n)) = F(k_n)$. Similarly, $s(F(k)) = F(k)$. We have thus shown that $F(k_n)$ converges to $F(k)$ in $\underline{S}$, and our proof is complete.

Remark 3.8. It is clear that the preceding lemma is valid under weaker assumptions about $\underline{C}$ and $\underline{K}$. For instance, it is sufficient to assume that, for every $\underline{k} \in \underline{K}$ and for every neighborhood $\underline{U}$ of $\underline{k}$, there exists a neighborhood $\underline{V}$ of $\underline{k}$ such that $\underline{V} \subset \underline{U}$ and $\underline{V} \cap \underline{C}$ is connected.

We now state and prove the theorem towards which we have been aiming.

THEOREM 3.9. Let $\underline{D} \subset \underline{V}(\underline{M})$ be a set of complete vector fields, and let $\underline{x} \in \underline{M}$. Then, for each $\underline{T} > 0$, $L_x(D,T) \subset I_0^t(D,x)$ and, moreover, the interior of $L_x(D,T)$(relative to $I_0^t(D,x)$) is dense in $L_x(D,T)$ (and is, in particular, non-empty).

Proof. If $\dim \mathscr{T}_0(D)(x) = \dim \mathscr{T}(D)(x)$; then we have from Corollary 3.4 that $\mathscr{T}_0(D)(y) = \mathscr{T}(D)(y)$ for all $y \in I(D,x)$. Therefore, $I_0(D,x) = I_0^t(D,x)$ and our conclusion follows from Theorem 3.3. Assume that $\dim \mathscr{T}_0(D)(x) = k = \dim \mathscr{T}(D)(x) - 1$. It is clear from Lemma 3.6 that, if α is an integral curve of $\underline{D}$ such that $\alpha(0) = x$, then $\alpha(T) \in I_0^T(D,x)$; hence, $L_x(D,T) \subset I_0^T(D,x)$.

We now show that, if $y \in L_x(D,T)$, then $\underline{y}$ is the limit of points which belong to the interior of $L_x(D,T)$. Let $D = \{X_1,\ldots,X_k\}$ and let $y = \Phi_{\underset{\sim}{i}}(\underset{\sim}{T},x)$, where $\|\underset{\sim}{T}\| = T$, and $T_i > 0$ for $i = 1,2,\ldots,m$ (the notations here are the same as in the proof of Theorem 3.1). Let $\underset{\sim}{j} = (j_1,\ldots,j_s)$ be an $\underline{s}$-tuple of integers between 1 and $\underline{k}$ such that the rank of $\underset{\sim}{t} \to \Phi_{\underset{\sim}{j}}(\underset{\sim}{t},x)$ is equal to $\dim \mathscr{T}(D)(x)$ for all $\underline{t}$ in an open dense subset Ω of $\underset{\sim}{R}^{\tilde{s}}$. Let $\Omega' = \{\underset{\sim}{t} : \underset{\sim}{t} \in \underset{\sim}{R}^s,\ t_i > 0 \text{ for } i = 1,\ldots,s\} \cap \Omega$. Let $\{\underset{\sim}{t}_p\} \subset \Omega$ be a sequence that converges to 0, and let $\underset{\sim}{T}_p = (T_1,\ldots,T_{m-1},T_m - \|\underset{\sim}{t}_p\|)$. We can assume that $\|t_p\| < T_m$ for all $p > 0$. If we let $y_p = \Phi_{\underset{\sim}{i}}(\underset{\sim}{T}_p, \Phi_j(\underset{\sim}{t}_p,x))$, then $y_p \in L_x(D,T)$. We next show that y_p is in the interior $\tilde{o}$f $L_x(D,T)$ relative to $I_0^T(D,x)$. Since the mapping $z \to \Phi_{\underset{\sim}{i}}(\underset{\sim}{T}_p,z)$ is a diffeomorphism from $I_0^{\|\underset{\sim}{t}_p\|}(D,x)$ onto $I_0^T(D,x)$, it suffices to show that $\Phi_{\underset{\sim}{j}}(\underset{\sim}{t}_p,$ ... s in the interior of $L_x(D,\|\underset{\sim}{t}_p\|)$. Let $V_p = \{\underset{\sim}{t} : \underset{\sim}{t} \in \underset{\sim}{R}^s,\ t_1 > 0,\ldots,t_s > 0,\ \|\underset{\sim}{t}\| = \|\underset{\sim}{t}_p\|\}$. Clearly, if $\underset{\sim}{t} \in V_p$, then $\Phi_j(\underset{\sim}{t},x) \in L_x(D,\|\underset{\sim}{t}_p\|)$. Let $F_p : V_p \to I_0^{\|\underset{\sim}{t}_p\|}(D,x)$ be defined by $F_p(\tilde{\underset{\sim}{t}}) = \Phi_{\underset{\sim}{j}}(\underset{\sim}{t},x)$. We show that F_p is analytic. Since F_p

is analytic as a map from V_p into $I(D,x)$, it suffices to show that it is continuous. But this follows from the previous lemma, because V_p is convex. The rank of $\underset{\sim}{t} \to \Phi_j(\underset{\sim}{t}, x)$ is equal to $\dim \mathscr{T}(D)(x)$ at $\underset{\sim}{t} = \underset{\sim}{t}_p$. Since V_p is a submanifold of $\underset{\sim}{R}^s$ of codimension 1, it follows that the rank of F_p at $\underset{\sim}{t}_p$ is equal to the dimension of $I_0^{\|\underset{\sim}{t}_p\|}(D,x)$. Thus, $F_p(V_p)$ contains a neighborhood of $F_p(\underset{\sim}{t}_p)$ in $I_0^{\|\underset{\sim}{t}_p\|}(D,x)$. It follows that $\underset{\sim}{\Phi}_j(\underset{\sim}{t}_p, x)$ is in the interior of $L_x(D, \|\underset{\sim}{t}_p\|)$. By the previous remark we conclude that y_p is interior to $L_x(D,T)$ in $I_0^T(D,x)$. There remains to be shown that y_p converges to y in $I_0^T(D,x)$. The mapping $(\underset{\sim}{t}, \underset{\sim}{s}) \to \underset{\sim}{\Phi}_i(\underset{\sim}{t}, \underset{\sim}{\Phi}_j(s,x))$ is continuous as a map from $\underset{\sim}{R}^m X \underset{\sim}{R}^s$ into $I(D,x)$. The set $V = \{(\underset{\sim}{t}, \underset{\sim}{s}) : t_i > 0,\ s_j > 0, i = 1, \dots, m,\ j = 1, \dots, s,\ \|\underset{\sim}{t}\| + \|s\| = T\}$ is convex, and is mapped into $I_0^T(D,x)$. Therefore, the previous lemma is applicable, and we conclude that $y_p \to y$ in $I_0^T(D,x)$. This proves our theorem.

4. Applications to Control Systems

We shall consider systems of the form

$$\frac{dx(t)}{dt} = F(x(t), u(t))$$

defined on an analytic manifold $\underline{M}$. The functions $\underline{u}$ belong to a class $\mathscr{U}$ of "admissible controls". We make the following assumptions about $\mathscr{U}$ and the system function $\underline{F}$:

(i) <u>The elements of $\mathscr{U}$ are piecewise continuous functions defined in</u> $[0,\infty)$, <u>having values in a locally path connected set</u> Ω. $\Omega \subset \underset{\sim}{R}^m$ (Ω is locally path connected if, for every $\omega \in \Omega$ and every neighborhood $\underline{V}$ of ω, there exists a neighborhood $\underline{U}$ of ω such that $\underline{U} \subset \underline{V}$, and $\underline{U} \cap \Omega$ is path connected). In addition, we assume that $\mathscr{U}$ <u>contains</u>

all the piecewise constant functions with values in Ω , and that every element of $\mathscr{U}$ has finite one-side limits in each point of discontinuity. $\mathscr{U}$ is endowed with the topology of uniform convergence on compact intervals.

(ii) $F: M \times \Omega \to TM$ is jointly continuously differentiable. For each $u \in \Omega$, $F(\cdot, u)$ is a complete analytic vector field on M. We know that for each $x \in M$, $u \in \mathscr{U}$, the differential equation

$$\frac{dx(t)}{dt} = F(x(t), u(t)) \qquad x(0) = x, \tag{1}$$

has a solution defined for all $t \in [0, \delta)$, where $\delta > 0$. We denote such a solution by $\pi(x, u, \cdot)$, and we assume that $\pi(x, u, t)$ is defined for all $t \in [0, \infty)$.

For the above defined control system, we now state the basic controllability concepts. We say that $y \in M$ is attainable from $x \in M$ at time t ($t \geq 0$), if there exists $u \in \mathscr{U}$ such that $\pi(x, u, t) = y$. For each $x \in M$, we let $A(x, t)$ denote the set of all points attainable from x at time t. If $0 \leq t < \infty$, we define $\underset{\sim}{A}(x, t) = \bigcup_{s \leq t} A(x, s)$ and $\underset{\sim}{A}(x) = \bigcup_{t \geq 0} \underset{\sim}{A}(x, s)$. We say that the system is controllable from x if $\underset{\sim}{A}(x) = M$, and that it is controllable if it is controllable from every $x \in M$. We say that the system has the accessibility property from x if $\underset{\sim}{A}(x)$ has a non-empty interior, and that is has the accessibility property if it has the accessibility property from every $x \in M$. Finally, we shall say that the system has the strong accessibility property from x if $A(x, t)$ has a non-empty interior for some $t > 0$, and that it has the strong accessibility property if it has the strong accessibility property from x for every $x \in M$.

For $\omega \in \Omega$, let $X_\omega = F(\cdot, \omega)$; from assumption (ii) it follows that X_ω is a complete analytic vector field on $\underline{M}$. Throughout the remaining part of this article we let $D = \{X_\omega : \omega \in \Omega\}$.

LEMMA 4.1. For each $\underline{x} \in \underline{M}$, $\underset{\sim}{A}(x)$ is contained in $I(D, x)$.

The proof is identical to that of Lemma 2.4, and will, therefore, be omitted.

Remark 4.2. It is easy to see that the control system defined by restricting $\underline{F}$ to $\underline{I}(\underline{D}, \underline{x})$ satisfies the same assumptions as the original system. Since it can be readily verified that the map $u \to \Pi(x, u, t)$ is continuous as a map from $\mathscr{U}$ into $\underline{M}$, it follows that this map is also continuous as a map from $\mathscr{U}$ into $I(D, x)$.

We now want to obtain a result for $\underline{A}(\underline{x}, \underline{t})$ which is similar to that of Lemma 4.1. It is here that the assumption about Ω will be utilized. Let $\mathscr{P}$ be the class of piecewise constant Ω-valued functions defined on $[0, \infty)$. Clearly, $\mathscr{P}$ is dense in $\mathscr{U}$. Moreover, the local connectedness of Ω implies that the condition of Remark 3.8 is satisfied (this can be easily verified, and we omit the proof). Thus, we can apply Lemma 3.7, with $C = \mathscr{P}$ and $K = \mathscr{U}$, to obtain the following result:

LEMMA 4.3. Let $\underline{x} \in \underline{M}$. For each $t \geq 0$, $A(x, t) \subset I_0^t(D, x)$.

Proof. Since $\mathscr{U}$ contains $\mathscr{P}$, we have that $L_x(D, t) \subset A(x, t)$. Let $G : \mathscr{U} \to I(D, x)$ be defined by $G(u) = \Pi(x, u, t)$. We have that $G(\mathscr{P}) = L_x(D, t)$ and by Theorem 3.9, $G(\mathscr{P}) \subset I_0^t(D, x)$. Now our conclusion follows immediately from Lemma 3.7, and the proof is complete.

The above lemmas combined with the theorems of the preceding section yield the following results:

THEOREM 4.4. Let $\underline{x} \in \underline{M}$. Then $\underset{\sim}{A}(x) \subset I(D, x)$. Moreover, for every $\underline{T} > 0$, the interior of $\underset{\sim}{A}(x, T)$ relative to $I(D, x)$ is dense in

$\underset{\sim}{A}(x, T)$(and, in particular, is non-empty).

Proof. The first part is just the statement of Lemma 4.1. To prove the second part, we can assume that $I(\dot{D}, x) = M$ (if not, replace the original system by its restriction to $I(D, x)$, cf. Remark 4.2). Since $\underset{\sim}{L}_x(D, T)$ is dense in $\underset{\sim}{A}(x, T)$, our conclusion follows immediately from Theorem 3.1.

THEOREM 4.5. Let $\underline{x} \in \underline{M}$. Then, for each $t > 0$, $A(x,t) \subset I_0^t(D, x)$ and, moreover, the interior of $A(x, t)$ relative to $I_0^t(D, x)$ is dense in $A(x, t)$ (and, in particular, is non-empty).

Proof. The first part is just the statement of Lemma 4.3. To prove the second part, we apply Lemma 3.7 to the function $\underline{G}$ of Lemma 4.3, and we get that $\underline{G}$ is continuous as a map into $I_0^t(D, x)$; therefore, $L_x(D, t)$ is dense in $A(x, t)$ relative to $I_0^t(D, x)$. Our conclusion now follows immediately from Theorem 3.9, and the proof is complete.

The following two controllability criteria follow immediately from the Theorems 4.4 and 4.5, and from Lemma 2.1:

COROLLARY 4.6. The system has the accessibility property from $\underline{x}$ if and only if $\dim \mathscr{T}(D)(x) = \dim M$. In this case $\underset{\sim}{A}(x, T)$ has a non-empty interior for every $T > 0$.

COROLLARY 4.7. The system has the strong accessibility property from $\underline{x}$ if and only if $\mathscr{T}_0(D)(x) = \dim M$. In this case $A(x, T)$ has a non-empty interior for every $T > 0$.

The preceding results can be utilized to derive relationships between accessibility and strong accessibility. Even though the latter property seems much stronger than the former, we show that, for a very large class of manifolds (including the spheres S^n for $n > 1$, and all compact semisimple Lie groups, but not $\underset{\sim}{R}^n$), it is in fact implied by it.

On the other hand for a still larger class of manifolds (including $\underset{\sim}{R}^n$) controlability (which trivially implies accessibility), is sufficient to guarantee strong accessibility (the fact that controllability implies that $\dim \mathscr{T}(D^*)(x) = n + 1$ for all $\underline{x}$ was proved by Elliott in [5]).

Consider a system on a connected n-dimensional analytic manifold $\underline{M}$, having the accessibility property but not having the strong accessibility property. Let $\underline{D}$ be the family of associated vector fields. By Corollary 4.6, $\dim \mathscr{T}(D)(x) = n$ for all $x \in M$. By Corollary 3.4 the number $\dim \mathscr{T}_0(D)(x)$ is independent of $\underline{x}$. Since this number is either $\underline{n}$ or $\underline{n} - 1$, Corollary 4.7 implies that $\dim \mathscr{T}_0(D)(x) = n - 1$ for all $x \in M$. Choose a fixed $X \in D$, and use Φ_t to denote the one-parameter group generated by X(i.e., for every $y \in M$, the integral curve of X that passes through y at $t = 0$ is the curve $t \to \Phi_t(y)$). Define a mapping F from the manifold $S \times \underset{\sim}{R}$ into M by

$$F(s,t) = \Phi_t(s).$$

One shows easily that $\underline{F}$ is a local diffeomorphism onto $\underline{M}$. Moreover, $S \times \underset{\sim}{R}$ is connected. In fact, we have (cf. [18], Ch. 2, for the definition of a covering projection):

LEMMA 4.8. The map $\underline{F}$ is a covering projection.

Before we prove Lemma 4.8, we show how the results mentioned above follow from it.

THEOREM 4.9. Let $\underline{M}$ be a manifold whose universal covering space (cf. [18]) is compact. Then every system having the accessibility property has the strong accessibility property.

Proof. If the universal covering space of $\underline{M}$ is compact, then every covering space of $\underline{M}$ is compact. If it were possible to have a system on $\underline{M}$ having the accessibility property but not the strong accessibility property, we could define, for such a system, $\underline{S}$ and $\underline{F}$ as above. It would follow that $S X \underset{\sim}{R}$ is compact, which is clearly a contradiction.

Remark. If $n > 1$, the sphere S^n is simply connected (and compact). Therefore Theorem 4.9 applies. Also, if $\underline{M}$ is a connected compact semisimple Lie group (for instance $SO(n)$, if $n > 2$), the universal covering group of $\underline{M}$ is also compact (cf. [7], p. 123) and, therefore, Theorem 4.9 applies in this case as well.

THEOREM 4.10. Let $\underline{M}$ be a manifold whose fundamental group has no elements of infinite order. Then every controllable system on $\underline{M}$ has the strong accessibility property.

Proof. A controllable system obviously has the accessibility property. Assume it does not have the strong accessibility property. Define $\underline{S}$ and $\underline{F}$ as before. We show that $\underline{F}$ is one-to-one. Otherwise, there would exist $s_0, s_0' \in S$ and a $T \neq 0$ such that $F(T, s_0') = \Phi_T(s_0') = F(0, s_0) = s_0$. Therefore $\Phi_T(S) = S$. Define $H: S X \underset{\sim}{R} \to S X \underset{\sim}{R}$ by $H(s,t) = (\Phi_T(s), t-T)$. Then $\underline{H}$ is well defined, because $\Phi_T(S) = S$, and is a homeomorphism. Moreover, if $(s,t) \in S X \underset{\sim}{R}$

$$F(H(x,t)) = \Phi_{t-T}(\Phi_T(s)) = \Phi_t(s) = F(s,t).$$

Therefore $\underline{H}$ is a covering transformation (cf. [18], Ch. 2). Moreover, H has infinite order, because $H^m(s,t) = (\Phi_{mT}(s), t-mT)$, so that H^m is not the identity map if $m \neq 0$. We know from [18] Ch. 2 that the group of covering transformations of the covering space $(S X \underset{\sim}{R}, F)$ is isomorphic to a subgroup of the fundamental group π of M.

If π has no elements of infinite order, then this is a contradiction. Therefore F must be one-to-one. On the other hand, the points that are attainable from x_0 must belong to $S_t (= \Phi_t(S))$ for some nonnegative $\underline{t}$ (cf. Theorem 4.5). Therefore the points in S_{-t} are not attainable, if $t > 0$. Thus, the system is not controllable, and we have reached a contradiction.

Remark. Theorem 4.10 applies, in particular, to any simply connected manifold, such as $\underset{\sim}{R}^n$.

Proof of Lemma 4.8. We must show that every point of $\underline{M}$ has a neighborhood that is evenly covered by $\underline{F}$. Let $m \in M$. Since $\underline{F}$ is a local diffeomorphism onto, there exist $\underline{s} \in \underline{S}$, $t \in \underset{\sim}{R}$, $\epsilon > 0$ and a connected neighborhood $\underline{U}$ of $\underline{s}$ in $\underline{S}$ such that $F(s, t) = m$ and that the restriction of F to $UX(t-\epsilon, t+\epsilon)$ is a diffeomorphism onto an open subset $\underline{V}$ of $\underline{M}$. We claim that $\underline{V}$ is evenly covered. Let $A = \{\tau : \Phi_\tau(S) = S\}$. For each $\tau \in A$, let $U_\tau = \Phi_\tau(U)$. Since $\Phi_\tau : S \to S$ is a diffeomorphism, it follows that U_τ is open in $\underline{S}$ and connected for each $\tau \in A$. We first show that, if $0 < |\tau - \eta| < 2\epsilon$, $\tau \in A$, $\eta \in A$, then U_τ and U_η are disjoint. Assume they are not. Then $\Phi_T(U_\tau)$ and $\Phi_T(U_\eta)$ are not disjoint, for any $\underline{T}$. Choose $\underline{T}$ such that both $\underline{T} + \tau$ and $\underline{T} + \eta$ belong to $(t-\epsilon, t+\epsilon)$. Let $\underline{u} = \Phi_{T+\tau}(u_1) = \Phi_{T+\eta}(u_2)$ be a common element, where u_1 and u_2 belong to $\underline{U}$. Then the points $(u_1 T + \tau)$ and $(u_2, T + \eta)$ belong to $UX(t-\epsilon, t+\epsilon)$. Since the restriction of $\underline{F}$ to this set is one-to-one, it follows that $\tau = \eta$, which is a contradiction. For each $\tau \in A$, let $W_\tau = U_\tau \times (t-\tau-\epsilon, t-\tau+\epsilon)$. We shall conclude our proof that $\underline{V}$ is evenly covered by showing:

(a) the sets W_τ are open, connected and pairwise disjoint,

(b) for each $\tau \in A$, $\underline{F}$ maps W_τ diffeomorphically onto $\underline{V}$, and

(c) the inverse image of $\underline{V}$ under $\underline{F}$ is the union of the sets W_τ.

The first two assertions of (a) are obvious. If τ and η belong to A, and $\tau \neq \eta$, then either $|\tau - \eta| < 2\varepsilon$ or $|\tau - \eta| \geq 2\varepsilon$. In the first case W_τ and W_η must be disjoint, because U_τ and U_η are disjoint. In the second case, W_τ and W_η are also disjoint, because the intervals $(t-\tau-\varepsilon, t-\tau+\varepsilon)$ and $(t-\eta-\varepsilon, t-\eta+\varepsilon)$ cannot have a point in common.

To prove (b), take $\tau \in A$. Define $G : U \times (t-\varepsilon, t+\varepsilon) \to W_\tau$ by $G(u,\sigma) = (\Phi_\tau(u), \sigma - \tau)$. Clearly, G is a diffeomorphism from $U \times (t-\varepsilon, t+\varepsilon)$ onto W_τ. Moreover if $u \in U$, $t - \varepsilon < \sigma < t + \varepsilon$, then $F(G(u,\sigma)) = \Phi_{\sigma-\tau}(\Phi_\tau(u)) = \Phi_\sigma(u) = F(u,\sigma)$. Since the restriction of $\underline{F}$ to $U \times (t-\varepsilon, t+\varepsilon)$ is a diffeomorphism onto $\underline{V}$, the same must be true for the restriction of $\underline{F}$ to W_τ.

Finally, we prove (c). Let $u \in S$, $\sigma \in \underset{\sim}{R}$ be such that $F(u,\sigma) \in \underline{V}$. Then there exist $u' \in U$, $\sigma' \in (t-\varepsilon, t+\varepsilon)$ such that $F(u',\sigma') = F(u,\sigma)$. Therefore $u = \Phi_{\sigma'-\sigma}(u')$. This implies, in particular, that $\tau = \sigma' - \sigma$ belongs to $\underline{A}$, and that $u \in U_\tau$. Moreover, since $t - \varepsilon < \sigma' < t + \varepsilon$, it follows that $t - \tau - \varepsilon < \sigma < t - \tau + \varepsilon$. Therefore $(u, \sigma) \in W_\tau$.

The proof Lemma 4.8 is now complete.

5. Examples.

Example 5.1. Let $M = \underset{\sim}{R}^n$, $\Omega = \underset{\sim}{R}^m$, and let $F : M \times \Omega \to \underline{\underline{TM}}$ be defined by $F(x,u) = Ax + Bu$, where $\underline{A}$ and $\underline{B}$ are, respectively, $\underline{n} \times \underline{n}$ and $\underline{n} \times \underline{m}$ real matrices. We have that $D = \{A(\cdot) + Bu : u \in \underset{\sim}{R}^m\}$. Let b_i denote the $\underline{i}$-th column of $\underline{B}$. Then, as shown by Lobry [16], $\mathscr{T}(D)(x)$ contains the vectors:

$$Ax \pm b_i\ ,\ \pm Ab_i, \dots, \pm A^{n-1}b_i \quad i = 1, \dots, m.$$

It is not difficult to see that the above set of vectors forms a system of generators for $\mathscr{T}(\underline{D})(\underline{x})$. From Corollary 4.6 we get that $\underset{\sim}{A}(\underset{\sim}{0}, t)$ has a non-empty interior in $\underset{\sim}{R}^n$ if and only if $\{\pm b_i, \pm Ab_i, \dots, \pm A^{n-1}b_i, i = 1, 2, \dots, m\}$ has rank $\underline{n}$; equivalently. $\underset{\sim}{A}(\underset{\sim}{0}, t)$ has a non-empty interior in $\underset{\sim}{R}^n$ if and only if rank $[B, AB, \dots, A^{n-1}B] = n$.

Since, obviously, $\mathscr{T}_0(D)(\underset{\sim}{0}) = \mathscr{T}(D)(\underset{\sim}{0})$, we conclude that $A(\underset{\sim}{0}, t)$ has a non-empty interior whenever $\underset{\sim}{A}(\underset{\sim}{0}, t)$ does. The above statements, along with the fact that $A(\underset{\sim}{0}, t)$ and $\underset{\sim}{A}(\underset{\sim}{0}, t)$ are linear subspaces of $\underset{\sim}{R}^n$, imply that, if rank $[B, AB, \dots, A^{n-1}B] = n$, then for each $t > 0$ $A(\underset{\sim}{0}, t) = \underset{\sim}{A}(\underset{\sim}{0}, t) = \underset{\sim}{A}(\underset{\sim}{0}) = \underset{\sim}{R}^n$ (Kalman [12]). Thus, in this example, the accessibility property is equivalent to controllability. This is, of course, not true in general.

Example 5.2. Let $M = \underset{\sim}{R}^n$, $\Omega = \{u \in \underset{\sim}{R}^m : 0 \le u_i \le 1,\ i = 1, \dots, m\}$, and let $F(x, u) = (A_0 + \Sigma_{i=1}^m A_i u_i)x$ for all $(\underline{x}, \underline{u}) \in \underset{\sim}{R}^n X\ \Omega$, where $A_0, \dots, A_m$ are nXn real matrices. Then $\underline{D}$ is the set of all vector fields X_u where $X_u(x) = (A_0 + \Sigma_{i=1}^m u_i A_i)\, x$. The set $\underset{\sim}{M}^n$ of all nXn real matrices is a Lie algebra, with the bracket defined by $[P, Q] = PQ - QP$. To each matrix $\underline{P}$ there corresponds a vector field $\underline{V}(\underline{P})$ defined by $V(P)(x) = Px$. It is easy to check that $V([P, Q]) = [V(P), V(Q)]$. Using this fact, the spaces $\mathscr{T}(D)(x)$ and $\mathscr{T}_0(D)(x)$ can be readily computed:

$$\mathscr{T}(D)(x) = \{Px : P \in \underset{\sim}{L}\},$$

and

$$\mathscr{T}_0(D)(x) = \{Px : P \in L\}$$

where $\underset{\sim}{L}$ is the Lie algebra spanned by $A_0, \ldots, A_m$, and $\underline{L}$ is the ideal of $\underset{\sim}{L}$ spanned by $A_1, \ldots, A_m$. We remark that for this example the theory of Section 4 is valid <u>even</u> if $\mathcal{U}$ <u>is the set of all bounded and measurable Ω-valued functions</u>. This is so because the only properties of the class of admissible controls that were utilized in Section 4 were: (a) that the class of piecewise constant controls is dense in $\mathcal{U}$ (in the topology of uniform convergence), and (b) that, if $\{\underline{u}_\alpha\}$ are elements of $\mathcal{U}$ that converge uniformly to $\underline{u}$, then $\Pi(\underline{u}_\alpha, \underline{x}, \underline{t})$ converges to $\Pi(\underline{u}, \underline{x}, \underline{t})$.

In our example, both (a) and (b) remain valid if the topology of uniform convergence is replaced by that of <u>weak</u> convergence. This is easy to verify, and we shall not do it here (see Kučera [14]). Moreover, the set of Ω-valued measurable functions defined in $[0, \underline{T}]$ is weakly compact. It follows that the sets $\underset{\sim}{A}(x,T)$, $A(x,T)$ are compact for each $\underline{T} > 0$. Denote their interiors (relative to $I(D,x)$ and $I_0^T(D,x)$ respectively) by Int $\underset{\sim}{A}(x,T)$, Int $A(x,T)$. It follows that $\underset{\sim}{A}(x,T)$ is the closure of Int $\underset{\sim}{A}(x,T)$, and that $A(x,T)$ is the closure of Int $A(x,T)$. Therefore, our results contain those of Kučera (in this connection, see also Sussmann [21]).

<u>Remark</u>. The result of the preceding example is a particular case of a more general situation. Let $\underline{G}$ be a Lie group, and let $\underline{M}$ be an analytic manifold on which $\underline{G}$ acts analytically to the left. Then there is a homomorphism λ from the Lie algebra of $\underline{G}$ into $\underline{V}(\underline{M})$, defined by

$$\lambda(X)(m) = \frac{d}{dt}\,(\exp(tX)\cdot m),$$

the derivative being evaluated at $\underline{t} = 0$. If $X_0, \ldots, X_k$ belong to the Lie algebra of $\underline{G}$, we can consider the control problem

$$\frac{dx}{dt} = X'_0(x) + \sum_{i=1}^{k} u_i X'_i(x),$$

where $X'_i = \lambda(X_i)$. Example 5.2 results by letting $G = GL(n, \underset{\sim}{R})$ and $\underline{M} = \underset{\sim}{R}^n$.

Example 5.3. This example shows that the analyticity assumptions are essential. Consider the following two systems defined in the $(\underline{x}, \underline{y})$ plane:

$$(S_1) \quad \dot{x} = f_1(x, y, u)$$
$$\dot{y} = g_1(x, y, u)$$

and

$$(S_2) \quad \dot{x} = f_2(x, y, u)$$
$$\dot{y} = g_2(x, y, u)$$

Let $f_1 \equiv f_2 \equiv 1$, $g_1 \equiv 0$, and $g_2(x, y, u) = \varphi(x)$ where φ is a C^∞ function which vanishes for $-\infty < \underline{x} < 1$, and which is equal to 1 for $\underline{x} > 2$. It is clear that for (S_1) the set $\underset{\sim}{A}((0,0))$ is the half line $\{(x, y) : y = 0, x \geq 0\}$ while, for (S_2), $\underset{\sim}{A}((0,0))$ has a non-empty interior. However, both systems are identical in a neighborhood of $(0,0)$.

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